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MODIFIED GURTIN'S VARIATIONAL PRINCIPLES IN THE LINEAR DYNAMIC THEORY OF VISCOELASTICITY

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Abstract—Using the Vainberg's theory of potential operators, variational principles are developed for linear dynamic theory of viscoelasticity. The Euler equations of the functional developed herein are the governing field equations, including the boundary and initial conditions as opposed to equivalent set of integro-differential equations of the Gurtin's method.

1. INTRODUCTION

A large class of *boundary-value* problems in engineering and mathematical physics can be replaced by equivalent variational statements which include all the features of the problem, such as the governing equations, boundary conditions, conditions of constraint and even jump conditions. Variational principles for *initial-value* problems have not been derived until the early 1960's. This is primarily due to the fact that the first order operators $(\partial/\partial t)$ encountered in initial-value problems, such as in heat conduction, are not self-adjoint with respect to bilinear forms of the type

$$\langle u_1, u_2\rangle = \int_\Omega \int_0^{t_0} u_1(\mathbf{x}, t)u_2(\mathbf{x}, t)\,\mathrm{d}x\,\mathrm{d}t. \tag{1.1}$$

Moreover, variational principles (for example, Hamilton's principle in dynamics) constructed using the bilinear form in (1.1) do not include the initial conditions of the problem. Rosen[1] developed *restricted variational principles* for unsteady-state heat transfer, in which the time derivative of the temperature is kept constant while varying the temperature. The Lagrangian methods advanced by Biot[2, 3] involve *quasi-variational principles* in which there are no variational integrals (i.e. the functionals are in varied form).

In 1963, Gurtin[4–6] introduced a noval approach to construct variational principles for linear initial-value problems. Gurtin's technique involves reducing the given initial-value problem to an equivalent boundary-value problem using the idea of convolutions. The resulting Euler–Lagrange equations are integro-differential equations equivalent to the original partial differential equations, and contain the initial conditions implicitly. The well celebrated papers of Gurtin[4–6] have led to numerous generalizations and applications[7–9].

Although the convolution technique of Gurtin permits variational formulation of linear initial-value problems, the resulting Euler equations are integro-differential equations. To transform these equations back to the partial differential equations, generally some approximate inversion technique must be used. This introduces additional error into variational methods of approximation.

The objective of the present paper is to present modified Gurtin's variational principles whose Euler equations are the governing differential equations of the problem as opposed to equivalent integro-differential equations of Gurtin's method. The basic step in the derivation of these variational principles is to choose a bilinear form with respect to which the operator associated with the problem is *potential*. Tonti[10] showed that the first order differential operator is potential with respect to a convolution bilinear form, and in a recent paper[11] variational principles for linear initial-value problems are derived based on Tonti's observation.

Following this introduction, certain preliminary definitions and notations are presented. Section 3 contains a review of certain elements of the Vainberg's variational theory[12, 13]. In Section 4 variational principles for the linear theory of viscoelasticity are presented.

2. SOME MATHEMATICAL PRELIMINARIES

Let Ω be an open bounded region in three-dimensional euclidean space E^3, and let $\partial\Omega$ be the smooth boundary of Ω. The closure of Ω is denoted by $\bar{\Omega} = \Omega \cup \partial\Omega$. Let $\partial\Omega_u$ and $\partial\Omega_\sigma$ denote disjoint sets whose union is $\partial\Omega$,

$$\partial\Omega_u \cup \partial\Omega_\sigma = \partial\Omega, \quad \partial\Omega_u \cap \partial\Omega_\sigma = \phi.$$

Further, let $\mathbf{n}$ be the unit outward normal to the boundary $\partial\Omega$, and let the symbol "X" denote the cartesian product of two sets.

We denote by $\mathbf{x} = (x_1, x_2, x_3)$ a material point in the body, and the time is denoted by t. For simplicity we assume that the material coordinates x_i are rectangular cartesian when the body occupies a reference configuration in E^3.

Convolution. Suppose that f and g are functions defined on $\Omega X[0, t_0]$, $t_0 < \infty$, with $f(\mathbf{x}),.)$ and $g(\mathbf{x},.)$ are continuous on $[0, t_0]$ for each $\mathbf{x} \in \Omega$. We define the convolution "$f*g$" of f and g by

$$(f*g)(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, \tau) g(\mathbf{x}, t-\tau)\, d\tau \tag{2.1}$$

which has the following properties:

(a) $f*g = g*f$ (commutativity)
(b) $f*g = 0$ implies either $f = 0$ or $g = 0$ (Titchmarsh's Theorem)
(c) $(f*g)*h = f*(g*h) = f*g*h$ (associativity)
(d) $f*(g+h) = f*g + f*h$ (distributivity) (2.2)

We now give some lemmas whose role in this paper is analogous to that of the fundamental lemma in the calculus of variations. Proofs of these lemmas can be found in[5].

2.1 *Lemma.* Let f be a sufficiently smooth function on $\Omega X[0, t_0]$ and suppose that

$$\int_\Omega (f*g)(\mathbf{x}, t)\, dx = 0, \quad t_0 < \infty \tag{2.3}$$

for every arbitrarily smooth function $g(\mathbf{x}, t) \in \Omega X[0, t_0]$ which, together with all its space derivatives, vanishes on $\partial\Omega X[0, t_0]$. Then

$$f(\mathbf{x}, t) = 0 \quad \text{in} \quad \bar{\Omega} X[0, t_0]. \tag{2.4}$$

2.2 *Lemma.* Let f be sufficiently smooth on $\partial\Omega_\sigma X[0, t_0]$ and suppose

$$\int_{\partial\Omega_\sigma} (f*g)(\mathbf{x}, t)\, dS = 0, \quad \mathbf{x} \in \partial\Omega_\sigma \tag{2.5}$$

for every arbitrarily smooth function $g(\mathbf{x}, t)$ which vanishes on $\partial\Omega_u X[0, t_0]$. Then

$$f(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \partial\Omega_\sigma X[0, t_0]. \tag{2.6}$$

2.3 *Lemma.* Let f_i be sufficiently smooth on $\partial\Omega_u X[0, t_0]$ and suppose

$$\int_{\partial\Omega_u} (f_i * g_{ij} n_j)(\mathbf{x}, t)\, dS = 0, \quad \mathbf{x} \in \partial\Omega_u \tag{2.7}$$

for every arbitrarily smooth symmetric tensor-valued function g_{ij} which, together with all its space derivatives, vanishes on $\partial\Omega_\sigma X[0, t_0]$. Then

$$f_i(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \partial\Omega_u\, X[0, t_0]. \tag{2.8}$$

2.4 *Lemma.* Let $f_i(\mathbf{x}, 0)$ be sufficiently smooth on $\bar{\Omega}$, and suppose

$$\int_\Omega f_i g_i \, \mathrm{d}\mathbf{x}, \quad \mathbf{x} \in \Omega \tag{2.9}$$

for every arbitrarily smooth function g_i vanishing at $t = 0$. Then

$$f_i(\mathbf{x}, 0) = 0 \quad \text{in} \quad \bar{\Omega}.$$

3. VARIATIONAL THEORY OF POTENTIAL OPERATORS

Consider a nonlinear operator equation of the form

$$\mathcal{N}[\mathbf{u}(\mathbf{x}, t)] = 0 \quad \text{in} \quad \bar{\Omega}\mathrm{X}[0, t_0] \tag{3.1}$$

where $\mathcal{N}$ is a nonlinear operator. Our primary concern here is to construct variational statements of (3.1). That is, we wish to construct a functional $J[u]$ which assumes a stationary value at the solutions of (3.1). Not all operator equations admit variational formulation. Here we study, following Vainberg [12, 13], sufficient conditions for an operator to admit variational formulation.

Variation of operators. Suppose that $\mathcal{N}$ is a nonlinear operator mapping a complete normed linear vector space (Banach space) $\mathcal{U}$ into another normed linear vector space $\mathcal{V}$. We shall denote the domain of the operator $\mathcal{N}$ by $\mathcal{D}(\mathcal{N})$, and the range by $\mathcal{R}(\mathcal{N})$. Then

$$\delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\eta}) \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}[\mathcal{N}(\mathbf{u} + \epsilon\boldsymbol{\eta})]|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{\mathcal{N}(\mathbf{u} + \epsilon\boldsymbol{\eta}) - \mathcal{N}(\mathbf{u})}{\epsilon} \tag{3.2}$$

is called the *Gateaux differential* (or variation) of $\mathcal{N}$ at $\mathbf{u}$ in the direction $\boldsymbol{\eta}$, provided the limit in (3.2) exists for any $\boldsymbol{\eta} \in \mathcal{D}(\mathcal{N})$. The differential $\delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\eta})$ is also, in general, a nonlinear operator (it is homogeneous in $\boldsymbol{\eta}$, but not always additive). Here we assume that the Gateaux differential is linear in $\boldsymbol{\eta}$. In this case, we can write

$$\delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\eta}) = \delta\mathcal{N}(\mathbf{u})\boldsymbol{\eta} \tag{3.3}$$

where $\delta\mathcal{N}(\mathbf{u})$ is called the *Gateaux derivative* of $\mathcal{N}(\mathbf{u})$ at $\mathbf{u}$.

Variation and gradient of a functional. Since a functional $J(\mathbf{u})$ on $\mathcal{U}$ is also an operator, we can speak of derivative (or variation) of J in the same sense as (3.2):

$$\delta_\eta J(\mathbf{u}, \boldsymbol{\eta}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[J(\mathbf{u} + \epsilon\boldsymbol{\eta}) - J(\mathbf{u})] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\mathbf{u} + \epsilon\boldsymbol{\eta})|_{\epsilon=0} \tag{3.4}$$

Assuming that there exists a linear Gateaux differential $\delta_\eta J(\mathbf{u}, \boldsymbol{\eta})$ at a point $\mathbf{u} \in \mathcal{D}(J)$, we can write

$$\delta_\eta J(\mathbf{u}, \boldsymbol{\eta}) = \delta J(\mathbf{u})\boldsymbol{\eta} \tag{3.5}$$

where $\delta J(\mathbf{u})$ is the (linear) Gateaux derivative of the functional J at $\mathbf{u}$. Since $\delta_\eta J(\mathbf{u}, \boldsymbol{\eta})$ is also a functional, $\delta J(\mathbf{u})$ is a linear functional with domain $\mathcal{D}(J)$.

If for fixed $\mathbf{u} \in \mathcal{D}(J)$ the derivative $\delta J(\mathbf{u})$ is a continuous linear functional, then it can be extended by continuity to a linear functional on all of $\mathcal{U}$. This extension is called the *gradient* of the functional J at $\mathbf{u}$, and $J(\mathbf{u})$ is called the *potential of the operator* $\mathcal{N}$ $(\mathbf{u})$.

$$\text{grad}\, J(\mathbf{u}) = \mathcal{N}(\mathbf{u}) \tag{3.6}$$

Then it follows from the definition of grad $J(\mathbf{u})$ that

$$\langle \text{grad}\, J(\mathbf{u}), \boldsymbol{\eta} \rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\mathbf{u} + \epsilon\boldsymbol{\eta})\Big|_{\epsilon=0} \tag{3.7}$$

where $\langle \mathbf{f}, \mathbf{u} \rangle$ denotes the value of the linear functional $\mathbf{f} \in \mathcal{U}'$ ($\mathcal{U}'$ being the dual of $\mathcal{U}$) at $\mathbf{u} \in \mathcal{U}$.

Potential operators. If an operator $\mathcal{N}$ from $\mathcal{D}(\mathcal{N}) \subset \mathcal{U}$ into $\mathcal{U}'$ is the gradient of some functional $J(\mathbf{u})$, $\mathcal{N}$ is called a *potential operator* and $J(\mathbf{u})$ is called its potential. A necessary and sufficient condition for an operator $\mathcal{N}$ to be potential is given in the following theorem proof of which can be found in[12].

3.1 *Theorem.* Let $\mathcal{N}$ be a continuous operator from $\mathcal{U}$ to $\mathcal{U}'$ which has a linear Gateaux differential $\delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\eta})$ at every point $\mathbf{u} \in \mathcal{D}(\mathcal{N})$. Then a necessary and sufficient condition that $\mathcal{N}$ be potential is that

$$\langle \delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\eta}), \boldsymbol{\xi} \rangle = \langle \delta_\eta \mathcal{N}(\mathbf{u}, \boldsymbol{\xi}), \boldsymbol{\eta} \rangle \tag{3.8}$$

That is, the linear Gateaux derivative of the operator $\mathcal{N}$ must be symmetric in $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$.

The condition (3.8) of symmetry of the derivative of the operator is necessary for the field to be conservative and for a functional to exist. It must be emphasized that the symmetry of an operator is relative to the bilinear form chosen. In other words an operator may not be potential with respect to one bilinear form and may be potential with respect to other. The potentiality of an operator can also be interpreted in an alternate way: if $\langle \mathcal{N}(\mathbf{u} + \epsilon\boldsymbol{\eta}), \boldsymbol{\eta} \rangle$ is continuous in ϵ, $0 \le \epsilon \le 1$, for any $\boldsymbol{\eta}$ is an open convex set $\omega \subset \mathcal{D}(\mathcal{N})$, then $\mathcal{N}(\mathbf{u})$ is potential if and only if, for any polygonal line $L \subset \omega$, the line integral

$$\int_L \langle \mathcal{N}(\mathbf{u}), \mathrm{d}\mathbf{u} \rangle \tag{3.9}$$

is independent of the path of integration (see, Vainberg [12, p. 56]).

Recall from variational calculus that vanishing of the first variation of a functional is a necessary condition for the functional to assume a stationary value. From (3.6) it is clear that $\mathcal{N}(\mathbf{u}) = 0$ is the Euler equation of the functional $J(\mathbf{u})$. Given a Gateaux differentiable functional $J(\mathbf{u})$, it is a simple matter to find its gradient. However, in parctice one is often faced with the opposite situation; namely, given an operator $\mathcal{N}$, find a functional such that (3.6) holds. Putting in other words, given the problem of solving a nonlinear equation of the form (3.1) by a variational method (such as the finite element method), find an associated functional. Note that, if $\mathcal{N}$ is a potential operator, from (3.9) it follows that the potential $J(\mathbf{u})$ of the operator $\mathcal{N}$ has the form

$$J(\mathbf{u}) = J(\mathbf{u}_0) + \int_{\mathbf{u}_0}^{\mathbf{u}} \langle \mathcal{N}(\mathbf{u}), \mathrm{d}\mathbf{u} \rangle$$

or alternatively,

$$J(\mathbf{u}_0) = J(\mathbf{u}_0) + \int_0^1 \langle \mathcal{N}(\mathbf{u}_0 + s(\mathbf{u} - \mathbf{u}_0)), \mathbf{u} - \mathbf{u}_0 \rangle \, \mathrm{d}s. \tag{3.10}$$

Thus, for equations represented by potential operators, there exists a functional (unique within a constant) such that (3.6) holds. This fact, proved by Vainberg[12], is stated in the following fundamental theorem.

3.2 *Theorem.* If $\mathcal{N}$ is a potential operator, then there exists a functional $J(u)$ whose gradient is the operator $\mathcal{N}$, and which is given by (3.10).

4. VARIATIONAL PRINCIPLES

Let u_i, γ_{ij}, σ_{ij}, f_i, G^{ijkl} and J_{ijkl} denote the components of the displacement vector $\mathbf{u}$, the Green's strain tensor $\boldsymbol{\gamma}$, the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$, the force vector $\mathbf{f}$, the relaxation tensor $\mathbf{G}$ and the compliance tensor $\mathbf{J}$, respectively. The equations governing linear theory of viscoelastic solids are

(i) Strain-displacement relations:

$$\gamma_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}] \quad \text{in} \quad \Omega \mathrm{X}[0, \infty] \tag{4.1}$$

(ii) Cauchy's equation of motion:

$$\sigma^{ij}_{,j}+\rho f_i=\rho\frac{\partial^2 u_i}{\partial t^2}$$

$$\sigma^{ij}=\sigma^{ji} \quad \text{in} \quad \Omega X[0,\infty] \tag{4.2}$$

(iii) Stress-strain relations:
(a) relaxation type

$$\sigma^{ij}(\mathbf{x},t)=E^{ijkl}(\mathbf{x})\gamma_{kl}(\mathbf{x},t)+\dot{G}^{ijkl}*\gamma_{kl} \tag{4.3a}$$

(b) creep type

$$\gamma_{ij}(\mathbf{x},t)=C_{ijkl}(\mathbf{x})\sigma^{kl}(\mathbf{x},t)+\dot{J}_{ijkl}*\sigma^{kl} \tag{4.3b}$$

wherein $E^{ijkl}(\mathbf{x})=G^{ijkl}(\mathbf{x},0)$, and $C_{ijkl}(\mathbf{x})=J_{ijkl}(\mathbf{x},0)$. Here it is understood that all the field variables are functions of $(\mathbf{x},t)\in\Omega X[0,\infty')$. To this set we add
(iv) boundary conditions:

$$u_i=\hat{u}_i, \quad \text{in} \quad \partial\Omega_u X[0,\infty) \tag{4.4}$$

$$T^i\equiv\sigma^{ij}n_j=\hat{T}^i \quad \text{on} \quad \partial\Omega_\sigma X[0,\infty) \tag{4.5}$$

(v) initial conditions:

$$u_i(\mathbf{x},0)=d_i(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega} \tag{4.6}$$

$$\frac{\partial u_i}{\partial t}(\mathbf{x},0)=v_i(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega} \tag{4.7}$$

where $\hat{u}_i$, $\hat{T}_i$, d_i and v_i are prescribed functions. The set (4.1)–(4.3a), (4.4)–(4.7) can be put into an operator form by setting

$$P=\begin{bmatrix} \rho\partial^2/\partial t^2 & 0 & -\frac{1}{2}\left(\delta_{im}\frac{\partial}{\partial x_j}+\delta_{jm}\frac{\partial}{\partial x_i}\right) & 0 & 0 & 0 & 0 \\ 0 & (\dot{G}^{ijkl}*+E^{ijkl})\delta_{ik}\delta_{jl} & -1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\left(\delta_{mj}\frac{\partial}{\partial x_i}+\delta_{mi}\frac{\partial}{\partial x_j}\right) & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 & -\rho & 0 \end{bmatrix} \tag{4.8}$$

$$\Lambda=\{u_m,\gamma_{ij},\sigma^{ij};\quad u_i,T^i;\quad u_i,\dot{u}_i\}^T \tag{4.9}$$

$$\Gamma=\{\rho f_m;\quad 0,0;\quad \hat{T}^i,-\hat{u}_i;\quad v_i,-d_i\}^T \tag{4.10}$$

Here we remark that the first three elements $(u_m, \gamma_{ij}, \sigma_{ij})$ in Λ are the basic dependent variables, and the remaining entries are the restrictions of the basic dependent variables to appropriate domains. Then (4.1)–(4.7) can be expressed in the operator form

$$\mathcal{N}(\Lambda)\equiv\mathcal{P}(\Lambda)-\Gamma=\mathbf{\Theta} \tag{4.11}$$

where $\mathbf{\Theta}=\{0,0,0;\ 0,0;\ 0,0\}^T$ is the zero vector.

We introduce the bilinear form

$$[f,g]=\int_\Omega\int_0^t f(\mathbf{x},\tau)g(\mathbf{x},t-\tau)\,\mathrm{d}x\,\mathrm{d}\tau \tag{4.12}$$

where $dx = dx_1 dx_2 dx_3$. We can show that $\partial/\partial t$ is self-adjoint with respect to the bilinear form in (4.12). Indeed, we have

$$
\begin{aligned}
\left[\frac{\partial f}{\partial t}, g\right] &= \int_\Omega \int_0^t \frac{\partial f}{\partial \tau}(\mathbf{x}, \tau) g(\mathbf{x}, t-\tau)\, dx\, d\tau \\
&= \int_\Omega \left[-\int_0^t f(\mathbf{x}, \tau) \frac{\partial}{\partial \tau} g(\mathbf{x}, t-\tau)\, dx \right. \\
&\quad \left. + f(\mathbf{x}, \tau) g(\mathbf{x}, t-\tau) \Big|_{\tau=0}^{t} \right] dx \\
&= \int_\Omega \int_0^t f(\mathbf{x}, \tau) \frac{\partial}{-\partial \tau} g(\mathbf{x}, t-\tau)\, dx\, d\tau + [f, g]_0 \\
&= \int_\Omega \int_0^t f(\mathbf{x}, \tau) \frac{\partial}{\partial (t-\tau)} g(\mathbf{x}, t-\tau)\, dx\, d\tau + [f, g]_0 \\
&= \left[f, \frac{\partial g}{\partial t}\right] + [f, g]_0
\end{aligned}
\tag{4.13}
$$

where

$$
[f, g]_0 = \int_\Omega \{f(\mathbf{x}, \tau) g(\mathbf{x}, t-\tau)\} \Big|_{\tau=0}^{t} dx \tag{4.14}
$$

Thus, $\partial/\partial t$ is self-adjoint; it can be easily verified that $\partial^2/\partial t^2$ is also self-adjoint. We also introduce the following notation for convenience:

$$
[f, g]_{\partial\Omega} = \int_{\partial\Omega} \int_0^t f(\mathbf{x}, \tau) g(\mathbf{x}, t-\tau)\, dS\, d\tau. \tag{4.15}
$$

In (4.15) f and g are the restrictions of functions f and g defined on $\Omega X[0, \infty)$ to the set $\partial\Omega X[0, \infty)$.

Let $\mathcal{S}$ denote the space of functions of the type $\Lambda = \{u_m, \gamma_{ij}, \sigma^{ij}; u_i, T^i; u_i, \dot{u}_i\}^T$. We define the bilinear form

$$
\begin{aligned}
\langle \Lambda_1, \Lambda_2 \rangle &= [u_m^1, u_m^2] + [\gamma_{ij}^1, \gamma_{ij}^2] + [\sigma_1^{ij}, \sigma_2^{ij}] \\
&\quad + [u_i^1, u_i^2]_{\partial\Omega_u} + [T_1^i, T_2^i]_{\partial\Omega_\sigma} + [u_i^1, u_i^2]_0 + [\dot{u}_i^1, \dot{u}_i^2]_0
\end{aligned}
\tag{4.16}
$$

wherein $\Lambda_\alpha = \{u_m^\alpha, \gamma_{ij}^\alpha, \sigma_\alpha^{ij}; u_i^\alpha, T_\alpha^i; u_i^\alpha, \dot{u}_i^\alpha\}^T$, $\alpha = 1, 2$. Substituting (4.11) into (3.10), and carrying out the indicated integration with respect to s, we obtain,

$$
\begin{aligned}
J(\Lambda) &= \frac{1}{2}[\rho \ddot{u}_m, u_m] - \frac{1}{2}[\sigma^{mj}_{,j}, u_m] - [\rho f_m, u_m] \\
&\quad + \frac{1}{2}[\dot{G}^{ijkl} * \gamma_{kl} + E^{ijkl}\gamma_{kl}, \gamma_{ij}] - [\sigma^{ij}, \gamma_{ij}] \\
&\quad + \frac{1}{2}\left[\frac{1}{2}(u_{i,j} + u_{j,i}), \sigma^{ij}\right] + \frac{1}{2}[T^m, u_m]_{\partial\Omega_\sigma} \\
&\quad - [\hat{T}^m, u_m]_{\partial\Omega_\sigma} - \frac{1}{2}[u_m, T^m]_{\partial\Omega_u} + [\hat{u}_m, T^m]_{\partial\Omega_u} \\
&\quad + \frac{1}{2}[\rho \dot{u}_m, u_m]_0 - [\rho v_m, u_m]_0 - \frac{1}{2}[\rho u_m, \dot{u}_m]_0 + [\rho d_m, \dot{u}_m]_0
\end{aligned}
\tag{4.17}
$$

Here the superimposed dot indicates differentiation with respect to time. Using integration by parts in time

$$
\begin{aligned}
\left[\rho \frac{\partial^2 u_m}{\partial t^2}, u_m\right] &= \int_\Omega \int_0^t \rho \frac{\partial^2 u_m}{\partial \tau^2}(\mathbf{x}, \tau) u_m(\mathbf{x}, t-\tau)\, dx\, d\tau \\
&= \int_\Omega \int_0^t \rho \frac{\partial u_m}{\partial \tau}(\mathbf{x}, \tau) \frac{\partial u_m}{\partial (t-\tau)}(\mathbf{x}, t-\tau)\, dx\, d\tau
\end{aligned}
$$

$$+\int_\Omega \left\{\rho\frac{\partial u_m}{\partial \tau}(\mathbf{x},\tau)u_m(\mathbf{x},t-\tau)\right\}\Bigg|_{\tau=0}^{t}\mathrm{d}x$$

$$\left[\rho\frac{\partial^2 u_m}{\partial t^2}, u_m\right] = \left[\rho\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial t}\right] + \left[\rho\frac{\partial u_m}{\partial t}, u_m\right]_0 \tag{4.18}$$

and the divergence theorem in space,

$$[\sigma^{mj}_{,j}, u_m] = -[\sigma^{mj}, u_{m,j}] + [T^m, u_m]_{\partial\Omega} \tag{4.19}$$

the functional $J(\Lambda)$ in (4.17) can be simplified to

$$\begin{aligned} J(\Lambda) = {} & \frac{1}{2}[\rho\dot{u}_m, \dot{u}_m] + \left[\sigma^{ij}, \frac{1}{2}(u_{i,j}+u_{j,i}) - \gamma_{ij}\right] - [\rho f_m, u_m] \\ & + \frac{1}{2}[\dot{G}^{ijkl} * \gamma_{kl} + E^{ijkl}\gamma_{kl}, \gamma_{ij}] - [T^m, u_m - \hat{u}_m]_{\partial\Omega_u} \\ & - [\hat{T}^m, u_m]_{\partial\Omega_\sigma} + [\rho(\dot{u}_m - v_m), u_m]_0 - \frac{1}{2}[\rho(u_m - 2d_m), \dot{u}_m]_0. \end{aligned} \tag{4.20}$$

Thus, (4.20) is a variational statement of the set in (4.1)–(4.7). That is, $J(\Lambda)$ assumes a stationary value when Λ is a solution of the set (4.1)–(4.7).

4.1 *Theorem.* Let $\Lambda = \{u_m, \gamma_{ij}, \sigma^{ij}; u_i, T^i; u_i, \dot{u}_i\}^T \in \mathcal{S}$ be the solution of (4.1)–(4.7). Suppose that we define, for each $(\mathbf{x}, t) \in \bar{\Omega}X[0, t_0]$, the functional $J(\cdot)$ on $\omega \subset \mathcal{S}$ by (4.20). Then the gradient of J satisfies the condition

$$\operatorname{grad} J(\Lambda) \equiv \delta J(\Lambda) = \mathbf{\Theta}, \quad \text{over} \quad \omega \subset \mathcal{S} \tag{4.21}$$

if and only if Λ is a solution of (4.1)–(4.7).

Proof. Let $\bar{\Lambda} = \{\bar{u}_m, \bar{\gamma}_{ij}, \bar{\sigma}^{ij}; \bar{u}_m, \bar{T}^m; \bar{u}_m, \bar{\dot{u}}_m\}^T$ be an arbitrary element in $\mathcal{S}$. Then the first variation (or Gauteax differential) of J is

$$\begin{aligned} \delta_{\bar{\Lambda}} J(\Lambda, \bar{\Lambda}) = {} & \frac{\mathrm{d}}{\mathrm{d}\alpha} J(\Lambda + \alpha\bar{\Lambda})|_{\alpha=0} \\ = {} & \left[\rho\frac{\partial u_m}{\partial t}, \frac{\partial \bar{u}_m}{\partial t}\right] + \left[\frac{1}{2}(u_{i,j}+u_{j,i}) - \gamma_{ij}, \bar{\sigma}^{ij}\right] \\ & + \left[\sigma^{ij}, \frac{1}{2}(\bar{u}_{i,j}+\bar{u}_{j,i}) - \bar{\gamma}_{ij}\right] - [\rho f_m, \bar{u}_m] \\ & + \frac{1}{2}[\dot{G}^{ijkl} * \gamma_{kl}, \bar{\gamma}_{ij}] + \frac{1}{2}[\dot{G}^{ijkl} * \bar{\gamma}_{kl}, \gamma_{ij}] \\ & - [u_m - \hat{u}_m, \bar{T}^m]_{\partial\Omega_u} - [\hat{T}^m, \bar{u}_m]_{\partial\Omega_\sigma} + [E^{ijkl}\gamma_{kl}, \bar{\gamma}_{ij}] \\ & + [\rho(\dot{u}_m - v_m), \bar{u}_m]_0 + [\rho\bar{\dot{u}}_m, u_m]_0 \\ & - \frac{1}{2}[\rho u_m - 2d_m, \bar{\dot{u}}_m]_0 - \frac{1}{2}[\rho\bar{u}_m, \dot{u}_m]_0. \end{aligned}$$

Integrating by parts with respect to time and using the divergence theorem, we can write

$$\begin{aligned} \delta_{\bar{\Lambda}} J(\Lambda, \bar{\Lambda}) = {} & \left[\frac{1}{2}(u_{i,j}+u_{j,i}) - \gamma_{ij}, \bar{\sigma}_{ij}\right] - [\sigma^{ij}_{,j} + \rho f_i - \rho\ddot{u}_i, \bar{u}_i] \\ & + [(\dot{G}^{ijkl} * \gamma_{kl} + E^{ijkl}\gamma_{kl} - \sigma_{ij}), \bar{\gamma}_{ij}] - [u_i - \hat{u}_i, \bar{T}^i]_{\partial\Omega_u} \\ & + [T^i - \hat{T}^i, \bar{u}_i]_{\partial\Omega_\sigma} + [\rho(\dot{u}_i - v_i), \bar{u}_i]_0 - [\rho(u_i - d_i), \bar{\dot{u}}]_0 \\ = {} & \langle \mathcal{P}(\Lambda) - \Gamma, \bar{\Lambda} \rangle. \end{aligned} \tag{4.22}$$

Note that $\delta_{\bar{\Lambda}} J(\Lambda, \bar{\Lambda})$ is linear in $\bar{\Lambda}$. Hence,

$$\delta\bar{\Lambda} J(\Lambda, \bar{\Lambda}) = \langle \delta J(\Lambda), \bar{\Lambda} \rangle. \tag{4.23}$$

We first prove sufficiency. Suppose that $\Lambda \in \mathcal{S}$ is the solution of (4.1)–(4.7). Then (4.22) becomes

$$\delta_\Lambda J(\Lambda, \bar{\Lambda}) = 0$$

which implies, in view of (4.23), (4.21).

To prove the necessity ("only if" part), assume that (4.21) holds. Then

$$\delta_{\bar{\Lambda}} J(\Lambda, \bar{\Lambda}) = 0$$

for all $\bar{\Lambda}$ such that $\Lambda + \alpha\bar{\Lambda} \in \omega$ for all real α. That is,

$$\begin{aligned}
&\int_\Omega \left\{\frac{1}{2}(u_{i,j} + u_{j,i}) - \gamma_{ij}\right\} * \bar{\sigma}_{ij}\, dx \\
&\quad - \int_\Omega \{\sigma^{ij}_{,j} + \rho f_i - \rho\ddot{u}_i\} * \bar{u}_i\, dx \\
&\quad + \int_\Omega \{(\dot{G}^{ijkl} * \gamma_{kl} + E^{ijkl}\gamma_{kl}) - \sigma^{ij}\} * \bar{\gamma}_{ij}\, dx \\
&\quad - \int_{\partial\Omega_u} \{u_i - \hat{u}_i\} * \bar{T}^i\, dS + \int_{\partial\Omega_\sigma} \{T^i - \hat{T}^i\} * \bar{u}_i\, dS \\
&\quad + \int_\Omega \rho\{\dot{u}_i - v_i\}\bar{u}_i \Big|_{\tau=0}^{t} dx - \int_\Omega \rho\{u_i - d_i\}\bar{\dot{u}}_i \Big|_{\tau=0}^{t} dx = 0.
\end{aligned}$$

In view of *Lemmas* 2.1–2.4, and the fact that the variations $\bar{u}(\mathbf{x}, 0)$ and $\bar{\dot{u}}(\mathbf{x}, 0)$ are zero, it follows that $\Lambda = \{u_m, \gamma_{ij}, \sigma^{ij};\ u_i, T^i;\ u_i, \dot{u}_i\}^T$ is the solution of (4.1)–(4.7). This completes the proof.

Comparing (4.22) and (4.23) we note that

$$\delta J(\Lambda) = \mathcal{P}(\Lambda) - \Gamma$$

which shows that $\mathcal{P}(\Lambda) - \Gamma \equiv \mathcal{N}(\Lambda)$ is potential.

From the functional in (4.20) we can derive some alternate variational principles. For instance, assume that the strain-displacement relations (4.1), the stress-strain relations (4.3a), and the displacement boundary conditions (4.4) are satisfied identically. Then we obtain from (4.20) a new functional

$$\begin{aligned}
J_1(\Lambda) = \frac{1}{2}[\rho\dot{u}_m, \dot{u}_m] - [\rho f_i, u_i] + \frac{1}{2}[\dot{G}^{ijkl} * \gamma_{kl} + E^{ijkl}\gamma_{kl}, \gamma_{ij}] \\
- [\hat{T}^i, u_i]_{\partial\Omega_\sigma} + [\rho(\dot{u}_i - v_i), u_i]_0 \\
- \frac{1}{2}[\rho(u_i - 2d_i), \dot{u}_i]_0
\end{aligned} \tag{4.24}$$

wherein $G^{ijkl} * \gamma_{kl} = \sigma^{ij}$ and $\gamma_{ij} = (1/2)\ (u_{i,j} + u_{j,i})$.

Instead of (4.3a), if the stress-strain relations of the creep type (4.3b) are used, a functional analogous to (4.20) can be constructed:

$$\begin{aligned}
K(\Lambda) = \frac{1}{2}[\rho\dot{u}_i, \dot{u}_i] - [\rho f_i, u_i] + [\sigma^{ij}, \gamma_{ij}] - \frac{1}{2}[C_{ijkl}\sigma^{kl}, \sigma^{ij}] \\
- \frac{1}{2}[\dot{J}_{ijkl} * \sigma^{kl}, \sigma^{ij}] - [u_i - \hat{u}_i, T^i]_{\partial\Omega_u} - [\hat{T}^i, u_i]_{\partial\Omega_\sigma} \\
+ [\rho(\dot{u}_i - v_i), u_i]_0 - \frac{1}{2}[\rho(u_i - 2d_i), \dot{u}_i]_0
\end{aligned} \tag{4.25}$$

Now if the equations of motion (4.2) and the traction boundary conditions (4.5) are assumed to be satisfied identically, we derive a new functional from (4.25):

$$K_1(\Lambda) = \frac{1}{2}[\dot{J}_{ijkl} * \sigma^{kl}, \sigma^{ij}] + \frac{1}{2}[C_{ijkl}\sigma^{kl}, \sigma^{ij}] - [\hat{u}_i, T^i]_{\partial\Omega_u}$$
$$+ \frac{1}{2}[\rho(\dot{u}_i - v_i), u_i]_0 + [\rho d_i, \dot{u}_i]_0. \tag{4.26}$$

For quasi-static motion of linear viscoelastic solids, the functional J of (4.20) takes the form

$$J_s(\Lambda) = \left[\sigma^{ij}, \frac{1}{2}(u_{i,j} + u_{j,i}) - \gamma_{ij}\right] - [\rho f_i, u_i] + \frac{1}{2}[E^{ijkl}\gamma_{kl}, \gamma_{ij}]$$
$$+ \frac{1}{2}[\dot{G}^{ijkl} * \gamma_{kl}, \gamma_{ij}] - [T^i, u_i - \hat{u}_i]_{\partial\Omega_u}$$
$$- [\hat{T}^m, u_m]_{\partial\Omega_\sigma}. \tag{4.27}$$

Similarly other variational principles can be derived from (4.20) and (4.25).

5. CONCLUSIONS

Using Tonti's[10] convolution bilinear form variational principles are constructed for the linear dynamic theory of viscoelasticity. A systematic procedure is also presented, following Vainberg[12, 13], for constructing variational principles for potential operator equations.The prime feature of the present method is the use of a convolution bilinear form, which does not require the Gurtin's transformation of the field equations into equivalent integro-differential equations. That is, the Euler equations of the functionals derived here are the original partial differential equations, the boundary conditions, and the initial conditions of the linear dynamic theory of viscoelasticity. For quasi-static case, the difference between the present method and the Gurtin's method disappears.

Although the present paper addresses to linear dynamic viscoelasticity equations only, the procedure given herein can be trivally extended to other initial-value problems. Variational principles for linear heat conduction are given elsewhere[11]. Application of the present method to elastostatics, piezoelectricity, hydroelasticity, thermoelasicity, elastodynamics, etc. can be carried very easily (see Oden and Reddy[14]). However, application of nonlinear initial- and boundary-value problems is yet to be investigated.

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